A student carrying a flu virus returns to an isolated college campus of 1000 students. The rate at which the virus spreads is proportional to the product of the number of infected students and the number of uninfected students. Let $x(t)$ be the number of infected students $t$ days after the student returns to campus. If no students enter or leave campus during the spread of the virus, the differential equation that models the virus transmission is $x' = kx(1000 - x)$. Find $x(t)$ if that first student is initially the only infected student, and there are a total of 50 infected students 4 days later. Your final answer may use the symbol $k$, as long as you determine the value of $k$.

**SCORE: _____ / 9 PTS**

$$\frac{dx}{dt} = kx(1000-x)$$

$$\frac{1}{x(1000-x)}\,dx = k\,dt$$

$$\frac{1}{1000}\int\left(\frac{1}{x} + \frac{1}{1000-x}\right)dx = kt + C$$

$$\frac{1}{1000}\left(\ln|x| - \ln|1000-x|\right) = kt + C \quad \left(\tfrac{1}{2}\right)$$

$$\ln\left|\frac{x}{1000-x}\right| = 1000kt + C$$

$$\frac{x}{1000-x} = Ce^{1000kt}$$

$$\frac{1000-x}{x} = Ce^{-1000kt}$$

$$\frac{1000}{x} - 1 = Ce^{-1000kt}$$

$$x = \frac{1000}{1+Ce^{-1000kt}}$$

$$x(0) = 1 = \frac{1000}{1+C} \longrightarrow C = 999$$

$$x = \frac{1000}{1+999e^{-1000kt}} \quad \left(\tfrac{1}{2}\right)$$

$$x(4) = 50 = \frac{1000}{1+999e^{-4000k}}$$

$$1 + 999e^{-4000k} = 20$$

$$k = -\frac{1}{4000}\ln\frac{19}{999}$$

$$= \frac{1}{4000}\ln\frac{999}{19}$$

EACH UNDERLINED ITEM IS WORTH 1 POINT UNLESS INDICATED OTHERWISE

THIS ANSWER MAKES k LOOK NEGATIVE, AND IS BAD STYLE

ALTERNATE SOLUTION USING BERNOULLI EQUATION TECHNIQUE

(GRADE AGAINST ONLY 1 SOLUTION)

$$\frac{dx}{dt} = kx(1000-x)$$

$$\frac{dx}{dt} - 1000kx = -kx^2 \quad \frac{1}{2}$$

$\frac{1}{2}$ $v = x^{-1} \longrightarrow x = v^{-1} \longrightarrow \frac{dx}{dt} = -v^{-2}\frac{dv}{dt}$

$$-v^{-2}\frac{dv}{dt} - 1000kv^{-1} = -kv^{-2} \quad \frac{1}{2}$$

$$\frac{dv}{dt} + 1000kv = k \quad \frac{1}{2}$$

$$\mu = e^{\int 1000k\,dt} = e^{1000kt} \quad \frac{1}{2}$$

$$e^{1000kt}\frac{dv}{dt} + 1000ke^{1000kt}v = ke^{1000kt} \quad \frac{1}{2}$$

CHECKPOINT: $\frac{d}{dt}e^{1000kt} = 1000ke^{1000kt}$ ✓ $\frac{1}{2}$

$$e^{1000kt}v = \int ke^{1000kt}\,dt + C$$

$$= \frac{1}{1000}e^{1000kt} + C, \quad \frac{1}{2}$$

$$v = \frac{1}{1000} + Ce^{-1000kt}, \quad \frac{1}{2}$$

$$x = \frac{1}{\frac{1}{1000} + Ce^{-1000kt}}$$

$$= \frac{1000}{1 + Ce^{-1000kt}}$$

$$x(0) = 1 = \frac{1000}{1+C} \longrightarrow C = 999$$

$$x = \frac{1000}{1+999e^{-1000kt}} \quad \frac{1}{2}$$

$$x(4) = 50 = \frac{1000}{1+999e^{-4000k}}$$

$$1+999e^{-4000k} = 20$$

$$k = -\frac{1}{4000}\ln\frac{19}{999}$$

$$= \frac{1}{4000}\ln\frac{999}{19}$$

THIS ANSWER MAKES k LOOK NEGATIVE, AND IS BAD STYLE

A 200 liter tank initially contains 100 liter of pure water. Brine containing 3 grams of salt per liter starts flowing into the tank at 4 liters per minute. At the same time, the well-mixed solution leaves the tank at 2 liters per minute. **SCORE: _____ / 21 PTS**

[a] Find the amount of salt in the tank $t$ minutes after the brine starts flowing into the tank.

13 POINTS **HINT: Simplify all fractions as soon as possible**.

$$\frac{dA}{dt} = 4\cdot 3 - 2\cdot\frac{A}{100+(4-2)t}, \quad A(0)=0$$

$$\frac{dA}{dt} = 12 - \frac{A}{50+t} \quad 2$$

$$\frac{dA}{dt} + \frac{A}{50+t} = 12 \qquad \mu = e^{\int \frac{1}{50+t}dt} = e^{\ln|50+t|} = 50+t$$

$$(50+t)\frac{dA}{dt} + A = 12(50+t) \qquad \text{CHECKPOINT: } \frac{d}{dt}(50+t) = 1 \checkmark$$

$$(50+t)A = \int 12(50+t)\,dt + C$$

$$= 6(50+t)^2 + C$$

$$A = 6(50+t) + C(50+t)^{-1}$$

$$A(0) = 0 = 300 + \frac{C}{50} \longrightarrow C = -15000$$

$$A(t) = 6(50+t) - \frac{15000}{50+t}$$

[b] Find the concentration of salt in the tank $t$ minutes after the brine starts flowing into the tank.

2 POINTS

$$C(t) = \frac{A(t)}{100+2t} = 3 - \frac{7500}{(50+t)^2} \quad 2$$

[c] When the volume of solution in the tank equals the volume of the tank, the tank starts to overflow.

2 POINTS Find the concentration of salt in the tank at the instant the tank starts to overflow.

$$100 + 2t = 200 \longrightarrow t = 50$$

$$C(50) = 3 - \frac{7500}{10000} = 2\tfrac{1}{4}$$

[d] When the tank starts to overflow, the well-mixed solution leaves the tank at the same rate that the brine enters the tank.

4 POINTS Write, **BUT DO NOT SOLVE**, an initial value problem for the amount of salt in the tank $t$ minutes after the tank starts to overflow.

$$\frac{dA}{dt} = 4\cdot 3 - 4\cdot\frac{A}{200}, \quad A(0) = \text{AMOUNT OF SALT IN TANK @ TIME OF OVERFLOW } (t=50 \text{ ABOVE})$$

$$\frac{dA}{dt} = 12 - \frac{A}{50} \quad 2 \qquad = 600 - \frac{15000}{100} \text{ OR } 2\tfrac{1}{4}\times 200$$

$$= 450$$